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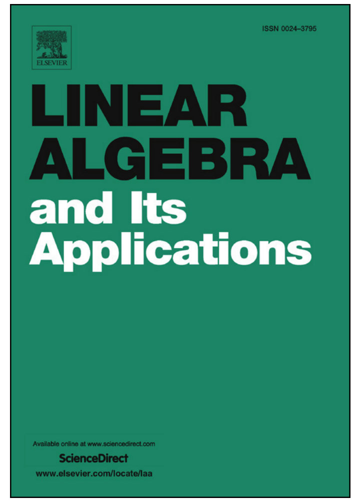

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PII: S0024-3795(20)30034-3
DOI: https://doi.org/10.1016/j.laa.2020.01.024
Reference: LAA 15269

To appear in: *Linear Algebra and its Applications*

Received date: 4 October 2019
Accepted date: 20 January 2020

Please cite this article as: P. Rowlinson, A problem concerning graphs with just three distinct eigenvalues, *Linear Algebra Appl.* (2020), doi: https://doi.org/10.1016/j.laa.2020.01.024.

Rowlinson P (2020) A problem concerning graphs with just three distinct eigenvalues. *Linear Algebra and its Applications*, 592, pp. 260-269. https://doi.org/10.1016/j.laa.2020.01.024

A PROBLEM CONCERNING GRAPHS WITH JUST THREE DISTINCT EIGENVALUES

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Abstract

We investigate the problem of finding all the biregrular graphs with just three adjacency eigenvalues, one of which is an eigenvalue $\neq -1, 0$ of maximal possible multiplicity.

AMS Classification: 05C50

Keywords: biregular graph, eigenvalue multiplicity, main eigenvalue, star complement, strongly regular graph.

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1 Introduction

We begin with some definitions. Let G be a graph of order n with $(0,1)$-adjacency matrix A. An eigenvalue σ of A is said to be an eigenvalue of G, and σ is a *main* eigenvalue if the eigenspace $\mathcal{E}_A(\sigma)$ is not orthogonal to the all-1 vector in $\mathbb{R}^n$. Always the largest eigenvalue, or *index*, of G is a main eigenvalue, and it is the only main eigenvalue if and only if G is regular. We say that G is *integral* if every eigenvalue of G is an integer. If G is connected with an eigenvalue $\neq -1, 0$ of multiplicity k and comultiplicity $t = n-k > 1$ then $k \leq \frac{1}{2}t(t-1)$ [1, Theorem 2.3]. If $k = \frac{1}{2}t(t-1)$ then we say that G has an eigenvalue $\neq -1, 0$ *of maximal multiplicity*.

A *biregular* graph is a graph with precisely two distinct degrees. A *strongly regular* graph, with parameters n, r, e, f, is an r-regular graph of order n in which any two adjacent vertices have e common neighbours, and any two non-adjacent vertices have f common neighbours. We take strongly regular graphs to include cliques and co-cliques. In accordance with [9] we say that G has a *strongly regular decomposition* if its vertex set has a bipartition $V(G) = X_1 \dot{\cup} X_2$ such that the subgraphs induced by X_1 and X_2 are strongly regular.

Let $\mathcal{B}$ be the class of connected biregular graphs with just three distinct eigenvalues, and let $\mathcal{C}$ be the class of connected graphs with an eigenvalue $\mu \neq -1, 0$ of maximal multiplicity. It is an open problem to determine the graphs in $\mathcal{B}$ [3], and another open problem, with origins in [1], to determine the graphs in $\mathcal{C}$. Graphs with just three distinct eigenvalues, subject to various restrictions, have recently been investigated in [3, 4, 12, 13].

It was noted in [12] that $\mathcal{B} = \mathcal{C}_1 \cap \mathcal{C}_2$, where $\mathcal{C}_1$ is the class of connected graphs with just three distinct eigenvalues, and $\mathcal{C}_2$ is the class of connected graphs with exactly two main eigenvalues; moreover any graph G in $\mathcal{B}$ is either integral or complete bipartite. Many examples are given in [3, 12, 13]. Since the spectrum of a bipartite graph is symmetric about 0, the only bipartite graph in $\mathcal{B} \cap \mathcal{C}$ is $K_{1,2}$. Thus any other graph in $\mathcal{B} \cap \mathcal{C}$ is non-bipartite and integral.

Apart from $K_{1,2}$, the only known graph in $\mathcal{C}$ is the graph Γ_{36} (of order 36) obtained from $L(K_9)$ by switching with respect to K_8; this is the largest exceptional graph, denoted by $G473$ in [6]. Since Γ_{36} has spectrum $-2^{(28)}, 5^{(7)}, 21$, the only known graphs in $\mathcal{B} \cap \mathcal{C}$ are $K_{1,2}$ and Γ_{36}. The problem investigated in this paper is to determine whether there are any other graphs in $\mathcal{B} \cap \mathcal{C}$. A solution to the problem remains elusive, but we do obtain two restrictive conditions on the graphs in question: in Proposition 4.2 we provide a relation between degrees and spectrum, and in Theorem 4.4 we show that such graphs have a strongly regular decomposition.

We use the notation of the monograph [7], where the basic properties of graph spectra can be found in Chapter 1. We write $\mathbf{j}$ for an all-1 vector, its length determined by context; and we write $u \sim v$ to indicate that vertices u, v of a graph are adjacent.

2 Graphs in $\mathcal{B}$

Here we note the results we require from [3] and [12]. Let G be a graph in $\mathcal{B}$ with degrees d_1, d_2, and let $X_i = \{v \in V(G) : \deg(v) = d_i\}$, $|X_i| = n_i$ $(i = 1, 2)$. Let the distinct eigenvalues of G be ρ, λ, μ, where ρ is the index of G. We have

$$(A - \lambda I)(A - \mu I) = \mathbf{v}\mathbf{v}^\top, \tag{1}$$

where $A\mathbf{v} = \rho\mathbf{v}$. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)^\top$: since G is connected, we may take $v_i > 0$ $(i = 1, 2, \ldots, n)$. It follows from Eq.(1) that $v_i^2 = d_1 + \lambda\mu$ if $i \in X_1$ and $v_i^2 = d_2 + \lambda\mu$ if $i \in X_2$. Hence $\mathbf{v} = \begin{pmatrix} a_1\mathbf{j} \\ a_2\mathbf{j} \end{pmatrix}$, where $a_i^2 = d_i + \lambda\mu$ $(i = 1, 2)$ and the partition of $\mathbf{v}$ is determined by $X_1 \dot\cup X_2$.

Lemma 2.1 *If $G \in \mathcal{B}$ then either*
(a) μ is non-main and $a_1 a_2 = -\lambda(\mu + 1)$, or
(b) λ is non-main and $a_1 a_2 = -\mu(\lambda + 1)$.

Proof. See [12, Lemma 2.2].

Lemma 2.2 *If $G \in \mathcal{B}$ then the sets X_1, X_2 form an equitable bipartition of G with divisor matrix $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, where*

$$k_{11} = \frac{a_1\rho - a_2 d_1}{a_1 - a_2}, \quad k_{12} = a_1\frac{d_1 - \rho}{a_1 - a_2}, \quad k_{21} = a_2\frac{d_2 - \rho}{a_2 - a_1}, \quad k_{22} = \frac{a_2\rho - a_1 d_2}{a_2 - a_1},$$

$$|X_1| = \frac{(\rho - a_2^2 + \lambda\mu)(\rho - \lambda)(\rho - \mu)}{(\rho + \lambda\mu + a_1a_2)(a_1^2 - a_1a_2)}, \quad |X_2| = \frac{(\rho - a_1^2 + \lambda\mu)(\rho - \lambda)(\rho - \mu)}{(\rho + \lambda\mu + a_1a_2)(a_2^2 - a_1a_2)}.$$

Proof. See [3, Theorem 4.3].

3 Graphs in $\mathcal{C}$

In this section we consider a graph G of order $n = \frac{1}{2}t(t+1)$ with an eigenvalue $\mu \neq -1, 0$ of multiplicity $k = \frac{1}{2}t(t-1)$ and comultiplicity $t > 1$. Let X be a star set for μ in G, that is, a set of k vertices such that μ is not an eigenvalue of $G - X$. The induced subgraph $H = G - X$ is called the corresponding star complement for μ (see [7, Chapter 5]). By [7, Proposition 5.1.4] the H-neighbourhoods of vertices in X are non-empty and distinct. If A_X is the adjacency matrix of the subgraph induced by X then G has adjacency matrix $A = \begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where C is the adjacency matrix of H. Then

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B \tag{2}$$

by [7, Theorem 5.1.7]. It follows that if $S = (B|C - \mu I) = (\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n)$ then $\mu I - A = S^\top(\mu I - C)^{-1}S$, and if $\langle\langle \mathbf{x}, \mathbf{y}\rangle\rangle$ denotes the bilinear form $\mathbf{x}^\top(\mu I - C)^{-1}\mathbf{y}$ ($\mathbf{x}, \mathbf{y} \in I\!R^t$) then

$$\langle\langle \mathbf{s}_u, \mathbf{s_v}\rangle\rangle = \begin{cases} -1 & \text{if } u \sim v \\ 0 & \text{if } u \not\sim v \text{ and } u \neq v \\ \mu & \text{if } u = v. \end{cases}$$

From [1] we know that the functions $\langle\langle \mathbf{s}_1, \mathbf{x}\rangle\rangle^2, \ldots, \langle\langle \mathbf{s}_n, \mathbf{x}\rangle\rangle^2$ form a basis for the space of homogeneous quadratic functions on $I\!R^t$. In particular, there exist $\alpha_1, \ldots, \alpha_n \in I\!R$ such that

$$\langle\langle \mathbf{x}, \mathbf{x}\rangle\rangle = \Sigma_{u=1}^n \alpha_u \langle\langle \mathbf{s}_u, \mathbf{x}\rangle\rangle^2. \tag{3}$$

Taking $\mathbf{x} = \mathbf{s}_v$ in Eq.(3) we have $\mu = \alpha_v\mu^2 + \Sigma_{u\sim v}\alpha_u$. Thus if $\mathbf{a} = (\alpha_1, \ldots, \alpha_n)^\top$ then

$$\mu\mathbf{j} = (\mu^2 I + A)\mathbf{a}. \tag{4}$$

It also follows from Eq.(3) that $\langle\langle \mathbf{x}, \mathbf{y}\rangle\rangle = \Sigma_{u=1}^n \alpha_u \langle\langle \mathbf{s}_u, \mathbf{x}\rangle\rangle\langle\langle \mathbf{s}_u, \mathbf{y}\rangle\rangle$; in particular, taking $\mathbf{y} = \mathbf{s}_v$ we have $\langle\langle \mathbf{s}_v, \mathbf{x}\rangle\rangle = \alpha_v\mu\langle\langle \mathbf{s}_v, \mathbf{x}\rangle\rangle - \Sigma_{u\sim v}\alpha_u$. Hence if $D = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ then

$$\begin{pmatrix} \langle\langle \mathbf{s}_1, \mathbf{x}\rangle\rangle \\ \langle\langle \mathbf{s}_2, \mathbf{x}\rangle\rangle \\ \vdots \\ \langle\langle \mathbf{s}_n, \mathbf{x}\rangle\rangle \end{pmatrix} = (\mu I - A)D \begin{pmatrix} \langle\langle \mathbf{s}_1, \mathbf{x}\rangle\rangle \\ \langle\langle \mathbf{s}_2, \mathbf{x}\rangle\rangle \\ \vdots \\ \langle\langle \mathbf{s}_n, \mathbf{x}\rangle\rangle \end{pmatrix}.$$

Taking $\mathbf{x} = \mathbf{s}_1, \ldots, \mathbf{s}_n$, we obtain:

Propostion 3.1 *If $G \in \mathcal{C}$ then, with the notation above,*

$$\mu I - A = (\mu I - A)D(\mu I - A). \tag{5}$$

4 Graphs in $\mathcal{B} \cap \mathcal{C}$

Here we retain the notation of Sections 2 and 3, and take G to be a graph in $\mathcal{B} \cap \mathcal{C}$ other than $K_{1,2}$. Let μ be the eigenvalue $\neq -1, 0$ of maximal multiplicity $\frac{1}{2}t(t-1)$. Note that $t > 2$ because G has order > 3. By [1, Theorem 3.1], μ is a main eigenvalue, and so the main eigenvalues of G are ρ and μ. It follows from Lemma 2.1 that $a_1a_2 = -\mu(\lambda+1)$, and it follows from [11, Proposition 2.1] that

$$(A - \rho I)(A - \mu I)\mathbf{j} = \mathbf{0}. \tag{6}$$

Lemma 4.1 *The matrix $\mu^2 I + A$ is invertible.*

Proof. Suppose by way of contradiction that $\lambda = -\mu^2$. Note first that if the eigenvalues $\mu_1, \ldots, \mu_t$ of G other than μ have mean d then $k\mu + td = 0$ and $k\mu^2 + \Sigma_{i=1}^t \mu_i^2 = n\bar{d}$, where $\bar{d}$ is the mean degree in G. We have

$$\Sigma_{i=1}^t(\mu_i - d)^2 = \Sigma_{i=1}^t \mu_i^2 - td^2 = (t+k)\bar{d} - \mu^2(k + t(\frac{k^2}{t^2})) = (t+k)\bar{d} - \mu^2 \frac{k}{t}(t+k),$$

and so $k\mu^2/t \leq \bar{d}$. Since $k = \frac{1}{2}t(t-1)$ and $\bar{d} < \rho$ we have $(t-1)\mu^2 < 2\rho$. Now $\rho = (t-1)\mu^2 - \frac{1}{2}t(t-1)\mu$, whence $\mu > 0$, for otherwise $-\mu \geq 2$ and we find that $\rho > 2n$. Since $\rho = \frac{1}{2}(t-1)(2\mu - t)\mu$ we have $\mu > \frac{1}{2}t$. Therefore $\frac{1}{4}t^2(t-1) < (t-1)\mu^2 < 2\rho \leq 2n - 2 = t(t+1) - 2$, whence $t \leq 5$ and $n \leq 15$. Now we have a contradiction either from [3, Table 1] or by calculating ρ, μ, n when $t = 5, 4, 3$. $\square$

In view of Lemma 4.1, we have $\mathbf{a} = \mu(\mu^2 I + A)^{-1}\mathbf{j}$ from Eq.(4). Now $\mu(\mu^2 I + A)^{-1}$ is a polynomial in A, while Eq.(6) shows that $\mathbf{j}$ is annihilated by a quadratic in A. It follows that $\mathbf{a} \in W$, where $W = \langle \mathbf{j}, A\mathbf{j} \rangle$, i.e. W is the subspace spanned by $\begin{pmatrix} \mathbf{j} \\ \mathbf{j} \end{pmatrix}$ and $\begin{pmatrix} d_1\mathbf{j} \\ d_2\mathbf{j} \end{pmatrix}$. Hence there exist $\alpha, \beta \in I\!R$ such that $\alpha_u = \alpha + \beta d_1$ if $u \in X_1$ and $\alpha_u = \alpha + \beta d_2$ if $u \in X_2$. Writing $\mathbf{d} = A\mathbf{j}$, we have from Eq.(4) and Eq.(6):

$$\mu\mathbf{j} = \mu^2\mathbf{a} + A\mathbf{a} = \mu^2\mathbf{a} + \alpha A\mathbf{j} + \beta A\mathbf{d} = \mu^2(\alpha\mathbf{j} + \beta\mathbf{d}) + \alpha\mathbf{d} + \beta A^2\mathbf{j}$$

$$= \mu^2(\alpha\mathbf{j} + \beta\mathbf{d}) + \alpha\mathbf{d} + \beta((\rho+\mu)\mathbf{d} - \rho\mu\mathbf{j} = (\mu^2\alpha - \beta\rho\mu)\mathbf{j}) + (\alpha + \mu^2\beta + \beta\mu + \beta\rho)\mathbf{d}.$$

It follows that

$$\alpha = \frac{\mu^2 + \mu + \rho}{(\mu+1)(\mu^2+\rho)}, \quad \beta = \frac{-1}{(\mu+1)(\mu^2+\rho)}. \tag{7}$$

Proposition 4.2 *If $G \in \mathcal{B} \cap \mathcal{C}$ then $(\rho - d_1)(d_2 - \rho) = (\mu+1)^2\rho$.*

Proof. Note that W $(= \langle \mathbf{j}, A\mathbf{j} \rangle)$ is D-invariant and A-invariant. We find the matrix of $D(\mu I - A)|_W$ with respect to the basis $\{\mathbf{j}, \mathbf{d}\}$. First, $D(\mu I - A)\mathbf{j} = D\begin{pmatrix} (\mu - d_1)\mathbf{j} \\ (\mu - d_2)\mathbf{j} \end{pmatrix} = \begin{pmatrix} w_1\mathbf{j} \\ w_2\mathbf{j} \end{pmatrix}$, where $w_i = (\alpha + \beta d_i)(\mu - d_i)$ $(i = 1, 2)$. Now $\begin{pmatrix} w_1\mathbf{j} \\ w_2\mathbf{j} \end{pmatrix} = p\begin{pmatrix} \mathbf{j} \\ \mathbf{j} \end{pmatrix} + q\begin{pmatrix} d_1\mathbf{j} \\ d_2\mathbf{j} \end{pmatrix}$, where $p = \dfrac{w_2 d_1 - w_1 d_2}{d_1 - d_2}$ and $q = \dfrac{w_1 - w_2}{d_1 - d_2}$. Secondly, $D(\mu I - A)\mathbf{d} = D(\rho\mu I - \rho A)\mathbf{j}$ by Eq.(6), and so the required matrix is $\begin{pmatrix} p & \rho p \\ q & \rho q \end{pmatrix}$.

Now Eq.(5) shows that $D(\mu I - A)|_W$ is idempotent and so $p + \rho q = 1$, i.e. $d_1 - d_2 = (\rho - d_2)(\alpha + \beta d_1)(\mu - d_1) - (\rho - d_1)(\alpha + \beta d_2)(\mu - d_2)$, or $1 = \beta(\rho - d_1)(\rho - d_2) - (\rho - \mu)(\rho\beta + \alpha)$. Substituting for α, β from Eq.(7), we obtain

$$\frac{(\rho - d_1)(\rho - d_2)}{(\mu+1)(\mu^2+\rho)} + \frac{(\rho - \mu)\mu}{\mu^2 + \rho} = -1,$$

and the result follows. $\square$

Example 4.3. For the graph Γ_{36} (with $d_1 < d_2$) we have $\mu = -2$, $\lambda = 5$, $\rho = 21$, $d_1 = 18$, $d_2 = 28$, $a_1 = 2\sqrt{2}$, $a_2 = 3\sqrt{2}$, $\alpha = -23/25$, $\beta = 1/25$.

For $i = 1, 2$, let G_i be the subgraph of G induced by X_i, with adjacency matrix A_i. In what follows, the neighbourhood of a vertex v in G is denoted by $\Delta(v)$. We also write $A = (a_{ij})$ and $A^2 = (a_{ij}^{(2)})$.

Theorem 4.4 *If $G \in \mathcal{B} \cap \mathcal{C}$ then the graph G_1 is strongly regular with parameters n_1, k_{11}, e_1, f_1, where*
$(d_1 - d_2)e_1 = (\mu^2 + \mu + \rho - d_2)\lambda(\mu + 1) + d_1\mu^2 + 3d_1\mu + d_1\rho - d_1d_2 - d_2\mu + \rho$,
$(d_1 - d_2)f_1 = (\mu^2 + \mu + \rho - d_2)(d_1 + \lambda\mu)$;
and the graph G_2 is strongly regular with parameters n_2, k_{22}, e_2, f_2, where
$(d_2 - d_1)e_2 = (\mu^2 + \mu + \rho - d_1)\lambda(\mu + 1) + d_2\mu^2 + 3d_2\mu + d_2\rho - d_1d_2 - d_1\mu + \rho$,
$(d_2 - d_1)f_2 = (\mu^2 + \mu + \rho - d_1)(d_2 + \lambda\mu)$.

Proof. From Eq.(3) we have

$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = (\alpha + \beta d_1)\Sigma_{u \in X_1} \langle\langle \mathbf{s}_u, \mathbf{x} \rangle\rangle \langle\langle \mathbf{s}_u, \mathbf{y} \rangle\rangle + (\alpha + \beta d_2)\Sigma_{u \in X_2} \langle\langle \mathbf{s}_u, \mathbf{x} \rangle\rangle \langle\langle \mathbf{s}_u, \mathbf{y} \rangle\rangle,$$

where α, β are given by Eq.(7). Hence if $i, j \in X_1$ and $i \sim j$ then

$$-1 = -2(\alpha + \beta d_1)\mu + (\alpha + \beta d_1)|\Delta(i) \cap \Delta(j) \cap X_1| + (\alpha + \beta d_2)|\Delta(i) \cap \Delta(j) \cap X_2|. \tag{8}$$

On the other hand, since $a_{ij}^{(2)} = a_1^2 + \lambda + \mu$ we have

$$d_1 + \lambda + \mu + \lambda\mu = |\Delta(i) \cap \Delta(j) \cap X_1| + |\Delta(i) \cap \Delta(j) \cap X_2|. \tag{9}$$

Now we can solve (8) and (9) for $|\Delta(i) \cap \Delta(j) \cap X_1|$ and $|\Delta(i) \cap \Delta(j) \cap X_2|$, to find that the number of common neighbours of i and j in X_1 is e_1, where $(d_1 - d_2)e_1$ is as given in the statement of the Proposition. Again if $i, j \in X_1$ and $i \not\sim j$ then

$$0 = (\alpha + \beta d_1)|\Delta(i) \cap \Delta(j) \cap X_1| + (\alpha + \beta d_2)|\Delta(i) \cap \Delta(j) \cap X_2|. \tag{10}$$

and

$$d_1 + \lambda\mu = |\Delta(i) \cap \Delta(j) \cap X_1| + |\Delta(i) \cap \Delta(j) \cap X_2|. \tag{11}$$

From (10) and (11),we find that the number of common neighbours of i and j in X_1 is f_1, where $(d_1 - d_2)f_1$ is as stated above. We may interchange X_1 and X_2 to obtain the second part of the Proposition. $\square$

Note that Theorem 4.4 says that G has a strongly regular decomposition in the sense of [9]. For Γ_{36} (with $d_1 < d_2$) we have $G_1 = L(K_8)$ with $e_1 = 6$, $f_1 = 4$, $k_{11} = 12$, and $G_2 = K_8$ with $e_2 = 6$, $f_2 = 9$, $k_{22} = 7$. (Although $f_2 > k_{22}$, this is not a contradiction because G_2 has no non-adjacent vertices, and $(A - 7I)(A + I)(A + 2I) = O$.)

Proposition 4.5 *Let G be a graph in $\mathcal{B} \cap \mathcal{C}$ other than $K_{1,2}$. If μ is not an eigenvalue of both G_1 and G_2 then $G = \Gamma_{36}$.*

Proof. Any non-main eigenvalue of G_i is a root of $x^2-(e_i-f_i)x-(k_{ii}-f_i)$ (cf. [7, Eq.(3.17)]), and one can check that $\mu^2-(e_i-f_i)\mu-(k_{ii}-f_i)=0$ for $i=1,2$. Thus if μ is not an eigenvalue of G_i then G_i is a clique or a co-clique because it has at most two distinct eigenvalues. In this case, by [10, Lemma 3] G_i is contained in a star complement for μ, and so $n_i \le t$. From the proof of Theorem 4.4 we see that for $i,j \notin X_i$ there are just two values for $|\Delta(i)\cap\Delta(j)\cap X_i|$ according as $i\sim j$, $i\not\sim j$. Therefore $n\le \frac{1}{2}n_i(n_i-1)$ by [5, Theorem 1.51]. Hence $n_i=t$ and G_i is a star complement for μ in G. If G_i is a clique then $G=\Gamma_{36}$ by [8, Theorem 5.1].

Now suppose for definiteness that G_2 is a co-clique. Then $k_{22}=0$ and $k_{21}=d_2$. From Eq.(2) we have $\mu I-A_1=\mu^{-1}B^\top B$, whence $k_{12}=\mu^2$. Moreover, if i,j are adjacent vertices in X_1 then $|\Delta(i)\cap\Delta(j)\cap X_2|=-\mu$; and if i,j are non-adjacent vertices in X_1 then $|\Delta(i)\cap\Delta(j)\cap X_2|=0$. Since $n_1\neq t$, the preceding argument shows that G_1 is not a clique; and G_1 is not a co-clique because G is not bipartite. It follows that $|\Delta(i)\cap\Delta(j)\cap X_2|$ $(i,j\in X_1)$ takes precisely two values, and so by [5, Theorem 1.51] the neighbourhoods $\Delta(i)\cap X_2$ $(i\in X_1)$ constitute a tight 4-design $\mathcal{D}$ (see [5, p.20]). Note that $\mu<0$, and so $\mu^2\ge 4$. Also, $\mu^2\le t-2$ because the neighbourhoods $\Delta(i)\cap X_2$ $(i\in X_1)$ are distinct. If $\mu^2<t-2$ then by [5, Theorem 1.54] $\mathcal{D}$ or its complement is the Steiner system $S(4,7,23)$. Therefore there are two cases to consider: (a) $\mu=-4$ and $t=23$, (b) $\mu^2=t-2$.

Now $\rho\mu=\det K=-\mu^2 d_2$, and so $\rho=-\mu d_2$. Since $k_{22}=0$, Lemma 2.2 shows that $a_1d_2=a_2\rho$, whence $a_1=-\mu a_2$ and $-\mu a_2^2=a_1a_2=-\mu(\lambda+1)$. Now we have

$$a_1^2=\mu^2(\lambda+1),\quad a_2^2=\lambda+1,\quad d_2=\lambda+1-\lambda\mu,\quad \rho=-\lambda\mu-\mu+\lambda\mu^2.$$

By Lemma 2.2,

$$t=n_2=\frac{(\rho+\lambda\mu-a_1^2)(\rho-\lambda)}{a_2^2-a_1a_2}=\frac{\mu(\mu d_2+\lambda)}{\lambda+1},$$

and so $t-1=d_2(\mu^2-1)/(\lambda+1)$. On the other hand, $\frac{1}{2}(t-1)=n_1/n_2=k_{21}/k_{12}=d_2/\mu^2$.

In case (a) we have $d_2=\frac{1}{2}(t-1)\mu^2=176$ and $\rho=704$, while $n=\frac{1}{2}t(t+1)=276$, a contradiction. In case (b) we have $2d_2=(t-1)\mu^2=\mu^2d_2(\mu^2-1)/(\lambda+1)$, whence $\lambda=\frac{1}{2}(t-1)(t-4)$ and $\rho=-\frac{1}{2}\mu(t-1)(t-2)$. Now $0=\mathrm{tr}A=\rho+\frac{1}{2}t(t-1)\mu+(t-1)\lambda$, whence $-2\mu=(t-1)(t-4)$ and $4(t-2)=4\mu^2=(t-1)^2(t-4)^2$, a contradiction. $\square$

It remains to consider the case in which μ is an eigenvalue of both G_1 and G_2. (A contradiction in this case would show that $\mathcal{B}\cap\mathcal{C}=\{K_{1,2},\Gamma_{36}\}$.)

5 The remaining case

Here we assume that G is a graph in $\mathcal{B}\cap\mathcal{C}$ other than Γ_{36}, so that μ is an eigenvalue of both G_1 and G_2. Moreover $\mu\neq -2$ for otherwise μ is the least eigenvalue of G and then Γ_{36} is the only candidate for G (see [6, Chapter 6]). We take the distinct eigenvalues of G_i to be k_{ii}, λ_i and μ. In this final section we offer four further observations which may be of interest to the reader who wishes to pursue matters further.

Remark 5.1 $|\mu+1| \leq |\sqrt{d_1}-\sqrt{d_2}|$.

Proof. Let $\kappa = (\mu+1)^2$. By Proposition 4.2, $\rho^2 = \rho(d_1+d_2-\kappa) - d_1d_2$, whence $\kappa < d_1+d_2$. The quadratic $\rho^2-(d_1+d_2-\kappa)\rho+d_1d_2$ has discriminant $(d_1+d_2-\kappa)^2-4d_1d_2$, and so $d_1+d_2-\kappa \geq 2\sqrt{d_1d_2}$, equivalently $\kappa \leq (\sqrt{d_1}-\sqrt{d_2})^2$ as required. □

Remark 5.2 $\lambda_1+\lambda_2 = \lambda+\mu$.

Proof. We have $\lambda_1+\mu = e_1-f_1$ and $\lambda_2+\mu = e_2-f_2$. By Theorem 4.4, $(\lambda_1+\lambda_2+2\mu)(d_2-d_1) =$

$\lambda(\mu^2+\mu+\rho-d_1)+2d_2\mu-d_1\mu+\rho-\lambda(\mu^2+\mu+\rho-d_2)-2d_1\mu-d_2\mu-\rho =$

$(d_2-d_1)(\lambda+3\mu)$, and the result follows. □

Remark 5.3 *If* $\mathbf{x}$ *is a* μ*-eigenvector of* G_1 *then* $\begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix}$ *is a* μ*-eigenvector of* G*, and if* $\mathbf{y}$ *is a* μ*-eigenvector of* G_2 *then* $\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}$ *is a* μ*-eigenvector of* G.

Proof. Let $A = \begin{pmatrix} A_1 & M_1^\top \\ M_1 & A_2 \end{pmatrix}$. From Eq.(1) we have

$$A_1^2 + M_1^\top M_1 - (\lambda+\mu)A_1 + \lambda\mu I = a_1^2 J.$$

Since $\mathbf{j}^\top\mathbf{x} = 0$, we have

$$\mu^2\mathbf{x} + M_1^\top M_1\mathbf{x} - (\lambda+\mu)\mu\mathbf{x} + \lambda\mu\mathbf{x} = \mathbf{0},$$

whence $M_1\mathbf{x} = \mathbf{0}$ and $A\begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix} = \mu\begin{pmatrix}\mathbf{x}\\ \mathbf{0}\end{pmatrix}$. Similarly, $A\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix} = \mu\begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}$. □

Remark 5.4 *If* $d_1 < d_2$ *and* $\mu < 0$ *then* $d_2 - \rho < \frac{a_1a_2+\mu-\lambda}{a_2^2}\rho < \rho$.

Proof. By Lemma 2.2 and Theorem 4.4 the inequality $f_2 \leq k_{22}$ becomes

$$\frac{(\mu^2+\mu+\rho-d_1)(d_2+\lambda\mu)}{d_1-d_2} \leq \frac{a_2\rho-a_1d_2}{a_2-a_1},$$

equivalently

$$d_2\mu + (\rho-d_1)\lambda + (\mu^2+\mu)\lambda \geq -\rho.$$

Since $\rho < d_2$, we have $(d_2+\mu\lambda)(\mu+1)+(\rho-d_1)\lambda > 0$. Multiplying by $\rho-d_2$, and invoking Proposition 4.2, we obtain

$$d_2-\rho < \frac{-(\mu+1)\lambda\rho}{d_2+\lambda\mu} = \frac{a_1a_2+\mu-\lambda}{a_2^2}\rho < \frac{a_1}{a_2}\rho < \rho.$$

□

Remark 5.2 is useful in ruling out various candidates for prescribed t and μ, as we illustrate below.

Example 5.5 Here we show that $(t, \mu) \neq (24, -3)$. If $t = 24$ then G_1, G_2 are strongly regular graphs whose orders sum to 300. Taking $n_1 \geq n_2$ without loss of generality, we find from Brouwer's list of feasible parameters [2] that when $\mu = -3$ the only possibilities are:
(i) $(n_1, k_{11}, \lambda_1) = (175, 102, 17)$, $(n_2, k_{22}, \lambda_2) = (125, 72, 12)$;
(ii) $(n_1, k_{11}, \lambda_1) = (231, 30, 9)$, $(n_2, k_{22}, \lambda_2) = (69, 20, 5)$;
(iii) $(n_1, k_{11}, \lambda_1) = (275, 162, 27)$, $(n_2, k_{22}, \lambda_2) = (25, 12, 2)$.
Now $\lambda = \lambda_1 + \lambda_2 + 3$ by Remark 5.2, and we can find ρ from the equation $\rho + (t-1)\lambda + \frac{1}{2}t(t-1)\mu = 0$. In each case we obtain the contradiction $\rho + \mu \neq k_{11} + k_{22}$. Analogous arguments eliminate the possibilities $(t, \mu) = (24, -4), (23, -3), (23, -4)$. $\square$

Acknowledgements The author is grateful to Jack Koolen for his interest in the problem, and to the anonymous referees for their constructive comments.

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